This exam consists of **two problems**, worth a total of 40 points.

Problem 1 has five parts, and is worth 25 points.
Problem 2 has three parts, and is worth 15 points.

Below are a few preliminaries and definitions that you may find useful for the exam.

- The Q-function is defined as

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right)\, dt,$$

  for $x \geq 0$.

- The following bound on the Q-function is valid:

$$Q(x) \quad \leq \quad \frac{1}{2} \exp\left(-\frac{x^2}{2}\right), \quad x \geq 0$$

1) Consider a bandpass transmission system that uses two signaling waveforms, given by

$$
\begin{aligned}
s_1(t) &= A_c k \sin(2\pi f_c t) + A_c \sqrt{1-k^2}\cos(2\pi f_c t) \quad \text{and} \\
s_2(t) &= A_c k \sin(2\pi f_c t) - A_c \sqrt{1-k^2}\cos(2\pi f_c t),
\end{aligned}
$$

for $0 \le t \le T_b$, where $A_c > 0$ and $k \in [0, 1)$ are parameters, and $f_c = n/T_b$ for some fixed integer n. (The first 'sin' term may be interpreted as a "carrier" component.)

a) [3 points] Find an orthonormal basis for these waveforms, and sketch a signal space diagram for this transmission scheme.

b) [4 points] Suppose that the communication system is coherent, and the received waveform is corrupted by additive white Gaussian noise with zero mean and power spectral density $N_0/2$. Find an expression for the average probability of (symbol) error, assuming that the signals are equally likely. Express your answer in terms of the Q-function, parameterized by k, N_0, and the energy per bit

$$E_b = \frac{1}{2}A_c^2 T_b.$$

c) [5 points] In the same setting as problem (b) above, suppose that k is chosen so that 10% of the transmitted signal power is allocated to the carrier component. Find a *sufficient* condition on the quantity E_b/N_0 to ensure that the probability of error does not exceed 10^{-4}. Use the upper bound on the Q-function provided on the previous page.

d) [6 points] Suppose that you want to modify the binary signaling scheme introduced above by adding one *additional* signaling waveform (and keeping the other two fixed), and that the new collection of waveforms will be equally likely. Specify this new waveform, under the conditions that the corresponding signal space representations for the new collection of 3 waveforms be equally spaced, and that the resulting constellation has the smallest average transmit power among all equally-spaced constellations. Draw and label the signal space diagram for this new ternary scheme.

e) [7 points] Given your set of 3 waveforms from part (d), specify the signal space coordinates for a new set of 3 equally-likely waveforms that has the same average probability of error in AWGN channels, but for which the average transmit power is minimum.

2) Suppose that you aim to transmit the signal

$$s(t) = 8\sin(3\pi t + \pi/2)$$

using 3-bit PCM.

a) [5 points] What is the Nyquist sampling rate for this signal?

b) [5 points] Suppose the sampling period is $T_s = 1/6$ seconds. Find the PCM waveform for the first 1/2 second.

c) [5 points] For $T_s = 1/6$ seconds, find the mean-square quantization error over the first 1/2 second.